Stochastic Modeling of Length of Day

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ABSTRACT

We propose a general stochastic model for the UT1-LOD system and derive the corresponding Kalman filter model. This stochastic model consists of an arbitrary sum of continous time autoregressive moving average (A RMA) processes, each chosen to characterize a different frequency band. The transition matrix which corresponds to the overall system and the time-dependent process noise covariance matrix are derived.

Based on the general formulation, several models for UT1 were derived from spectral analysis of the Space 92 UT1 series (Gross, 1993). Using Space 92 as the reference series, the candidate models were compared based on their ability to predict UT1 and LOD upto 30 days in the absence of data. These candidate models were compared with the JPL operational Kalman Earth Orientation Flter (KEOF) which assumes a random walk model for LOD (Morobito et al., 1987). The results of the comparison revealed that autoregressive modeling the 30-60 day oscillation in the LOD reduces the LOD prediction error by 10% to 20% during the first 30 days of prediction.

1 Introduction

The rotational rate of the solid earth (length of day) is not constant but undergoes small and unpredictable variations of up to several milliseconds in the length of day (LOD). Changes which occur over time scales of less than two years are dominated by the exchange of angular momentum between the atmosphere and the solid earth (crust and mantle), while decadal fluctuations have been attributed to the angular momentum exchange between the earth's liquid core and solid mantle (Hide and Dickey, 1991). Using a variety of space-age techniques, highly accurate observations of these changes have been made for over two decades. These include very long baseline interferometry (V LBI), satellite laser ranging (SLR), lunar laser ranging (L LR), and more recently global positioning system (GPS). Combining these measurements using the Kalman filter was first proposed by Morobito et al. (1987). In general, the Kalman filter combines a dynamic model (e.g. autoregressive time series model) and raw, irregularly spaced multidimensional observations (e.g. UT1 and polar motion) to produce predictions and smoothing of the observed time series. Stochastic models for the UT1 and polar motion were derived from analysis of atmospheric angular momentum (AAM) data (Morobito et al., 1987). A more recent description of Kalman filtering of earth orientation data is contained in Freedman et al. (1994).

The purpose of this paper is to continue and improve the stochastic modeling of UT1 and LOD. In particular, we will propose a general stochastic model for UT1 and derive the corresponding Kalman filter model. This stochastic model assumes that the UT1 is driven by an arbitrary sum of general autoregressive moving average (ARMA) processes, each characterizing a different frequency band. This general formulation of the Kalman filter allows for experimentation and flexibility in studying different stochastic models for UT1 and LOD, which in turn depends on the frequency band one wishes to model accurately. The use of ARMA processes as building blocks for the general stochastic processes is common in many applications (Brzezinski; Gelb; Haykin). Although each ARMA process is linear, certain linear combinations have been used to model nonlinear random processes, including those with power spectrums exhibiting $1/f$ power laws (Kushner, 1982).

We will derive several models for UT1 based on spectral analysis of the Space 92 LOD series (Gross, 1993). Using Space 92 as the reference series, the candidate models will be compared based on their ability to predict UT1 and LOD up to 30 days in the absence of data. These candidate models will be compared with the operational Kalman Earth Orientation filter (KEOF) which assumes a random walk model for LOD (Morobito et al., 1987).

2 Stochastic UT1 model

We will begin by formally defining a continous time ARMA(n,m) process. An autoregressive moving average process $Y(t)$ is a process which satisfies the stochastic differential equation

$$\frac{d^n}{dt^n}Y(t) + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}Y(t) + \ldots + a_1\frac{d}{dt}Y(t) + a_0Y(t) =$$

$$b_m\frac{d^m}{dt^m}W(t) + b_{m-1}\frac{d^{m-1}}{dt^{m-1}}W(t) + \ldots + b_1\frac{d}{dt}W(t) + b_0W(t), \tag{1}$$

where $W(t)$ is a continuous time white noise process with known excitation power q. The mathematical theory of such processes can be found in Priestly (1981). We will consider the case of $m = n - 1$, which implies that the process is aymptotically stable (Priestly, 1981). In state space form, equation (1) can be reduced to the first order system

$$\dot{X}(t) = AX(t) + BW(t) \tag{2}$$

$$Y(t) = CX(t), \tag{3}$$

where, in general, the matrices A, B, and C are $(n \times n)$, $(n \times 1)$, $(1 \times n)$ respectively, and the vector $x(t)$ is given by $X^T(t) = X_1(t), X_2(t), \dots X_n(t))$. There are infinitely many choices for the matrices in (2)-(3). We will use the representation (Faddeeva, 1959)

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \dots \\ 0 & 0 & 1 & 0 \dots \\ \vdots & & \ddots & \\ 0 & 0 & \dots & 1 \\ \vdots & & & \ddots \\ & & & 1 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \quad C^T = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix}, \tag{4}$$

The constant parameters a_i reflect the autoregressive part of the process, while the constant parameters b_i reflect the moving average part. Each process is fully determined by these two sets of parameters and the excitation power q.

The equation relating UT1 with the LOD is given by

$$\dot{U}(t) = -L(t) = -\frac{\Delta\Lambda(t)}{\Lambda_0}, \tag{5}$$

where $U(t)$ denotes the UT1, $\Delta\Lambda(t)$ denotes the variability in LOD, and $\Lambda_0 = 86400$ secs. $L(t)$ is assumed to satisfy the differential equation

$$\dot{L}(t) = Y_1 + Y_2 + \dots + Y_m + W_L, \tag{6}$$

where each Y_i is an ARMA(n,n-1) process with excitation power parameter q_i, and W_L is a white noise, independent of each Y_i, with excitation power q_L. Using the first order system representation of equations (2)-(3), we obtain the following matrix model for the UT1:

$$\begin{bmatrix} \dot{U} \\ \dot{L} \\ \dot{X}_1 \\ \dot{X}_2 \\ \vdots \\ \dot{X}_m \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & \dots & \dots & 0 \\ 0 & 0 & C_1 & C_2 & \dots & C_m \\ 0 & 0 & A_1 & 0 & \dots & 0 \\ \vdots & & & A_2 & & \\ \vdots & & & & \ddots & \\ 0 & \dots & \dots & \dots & \dots & A_m \end{bmatrix} \begin{bmatrix} U \\ L \\ X_1 \\ X_2 \\ \vdots \\ X_m \end{bmatrix} + \begin{bmatrix} 0 & 0 & \dots & \dots & \dots \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & B_1 & 0 & \dots & 0 \\ & & & B_2 & & \\ \vdots & & & & \ddots & \\ 0 & & & & & B_m \end{bmatrix} \begin{bmatrix} 0 \\ W_L \\ W_1 \\ W_2 \\ \vdots \\ W_m \end{bmatrix} \tag{7}$$

which will be abbreviated simply as

$$\dot{y}(t) = Fy(t) + Gv(t), \tag{8}$$

where the zero mean white noise vector $W'(t)$ has power spectral matrix Q_W.

We will also consider the LOD model

$$L(t) = Y_1 + Y_2 + \ldots + Y_m + W_L, \tag{9}$$

The resulting first order system representation can be obtained from equation (8) by deleting the first row and first column of matrices F and G, and deleting component L from the vector $\mathcal{Y}$. We will show that a similar result will hold for the corresponding Kalman filter equations. Thus, the model (9) for LOD is subsumed by the model given in equation (6).

The general solution to equation (8) is given by

$$\mathcal{Y}(t) = \Phi(t)\mathcal{Y}(0) + \int_0^t \Phi(t-s)G\mathcal{W}(s)ds, \tag{10}$$

where $\Phi(t)$ denotes the state transition matrix. The discrete time system corresponding to equation (10) is obtained by sampling $\mathcal{Y}(t)$ at times $\{t_k\}$, for $k \geq 0$. The resulting discrete time system is given by

$$\mathcal{Y}_{t_{k+1}} = \Phi(t_{k+1} - t_k)\mathcal{Y}_{t_k} + U_k, \tag{11}$$

where $\mathcal{Y}_{t_k} = \mathcal{Y}(t_k)$, and U_k is a zero mean vector given by

$$U_k = \int_{t_k}^{t_{k+1}} \Phi(t_{k+1} - s)\,G\mathcal{W}(s)ds, \tag{12}$$

with covariance matrix Q_k given by

$$Q_k = \int_{t_k}^{t_{k+1}} \Phi(t_{k+1} - s)GQ_WG^T\Phi^T(t_{k+1} - s)ds. \tag{13}$$

Below is a formula for $\Phi(t)$, whose derivation is detailed in Appendix A. Based on this formula, we derive the resulting form of Q_k in appendix B. Assume that the LOD is modeled as in equation (6), and the resulting system for the UT1 and LOD satisfies equation (8) with F given by (7). Then the transition matrix $\Phi(t) = e^{Ft}$ is given by

$$e^{Ft} = \begin{bmatrix} 1 & -t & Z_1 & Z_2 & \cdots & Z_m \\ 0 & 1 & R_1 & R_2 & \cdots & R_m \\ & & e^{A_1t} & & & \\ & & & e^{A_2t} & & \\ & & & & \ddots & \\ & & & & & e^{A_mt} \end{bmatrix} \tag{14}$$

where the vectors Z_i and R_i depend on the coefficients of the matrix A_i (recall that A_i is the ARMA system matrix which has the form given in equation (4)).

We will consider two cases, which correspond to two different classes of ARMA processes.

Case 1: A_i invertible : In this case, we have

$$Z_i(t) = C_iA_i^{-2}\left[I + tA_i - e^{A_it}\right], \tag{15}$$

$$R_i(t) = C_i A_i^{-1} \left[e^{A_i t} - I\right] \tag{16}$$

Case 2: $Y_i(t)$ satisfies the equation $\frac{d^n Y_i}{dt}(t) = b_0 W_i(t)$: In this case, $Z_i(t)$ and $R_i(t)$ reduce to

$$Z_i(t) = -b_0 \left(\frac{t^2}{2}, \frac{t^3}{3!}, \ldots, \frac{t^{n+1}}{(n+1)!}\right)' \tag{17}$$

$$R_i(t) = b_0 \left(t, \frac{t^2}{2}, \ldots, \frac{t^n}{n!}\right). \tag{18}$$

It is easily shown that $det(A_i) = (-1)^{n-1}(-a_0)$. Thus, case 1 holds if $a_0 \neq 0$. The transfer function corresponding to the differential equation has no poles at zero. In case 2, the ARMA coefficients are all equal to zero except for the coefficient b_0. For $n = 1$, case 2 corresponds to a random walk model, whereas for $n = 2$ it corresponds to an integrated random walk, The transfer function corresponding to case 2 has poles only at zero. The above formulation does not treat the case where the transfer function of the differential equation has simultaneous zero and nonzero poles. In this case, no closed form expression for $\Phi(t)$ was obtained, although other approximations for $\Phi(t)$ can be considered.

For A_i as in equation (4), the exponential matrix $e^{A_i t}$ which appears in equation (14) is well-understood. In most applications the matrix is computed numerically, However, in order to carry out the integration which appears in equation (13), it is desirable to have an explicit representation of $e^{A_i t}$ as a function of time. If the matrix A_i has simple eigenvalues (ie. of multiplicity one) given by $\{\lambda_1, \ldots, \lambda_n\}$, then $e^{A_i t}$ is given by $e^{A_i t} = V(t)[V^{-1}(0)]$, where $V(t)$ is given by

$$\begin{bmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} & \cdots & e^{\lambda_n t} \\ \lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_2 t} & \cdots & \lambda_n e^{\lambda_n t} \\ \vdots & & & \vdots \\ \lambda_1^{n-1} e^{\lambda_1 t} & \cdots & \cdots & \lambda_n^{n-1} e^{\lambda_n t} \end{bmatrix} \tag{19}$$

Explicit representation for $e^{A_i t}$ for the more complicated case of eigenvalues with multiplicity greater than one is derived in Hamdan (1993). For the second case where the process $Y_i(t)$ satisfies $\frac{d^n Y_i}{dt}(t) = b_0 W_i(t)$, the corresponding matrix $e^{A_i t}$ is given by

$$\begin{bmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{3!} & & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{n-2}}{(n-2)!} \\ 0 & 0 & 1 & & & \\ & & & & & \\ \vdots & & & & & t \\ 0 & & & & & 1 \end{bmatrix} \tag{20}$$

Finally, we note that for $L(t)$ given by equation (9), the corresponding transition matrix $\Phi(t)$ is obtained from the matrix of equation (14) by deleting the first row and first column.

In the next section, we will apply the above formulation to specific autoregressive models of LOD obtained by analyzing the power spectrum of Space 92 LOD data.

3 Examples of LOD models

One of the most popular methods for constructing ARMA models is based on spectral analysis of the observed times series. The frequency domain characterization of ARMA models is detailed in Priestly (1981). In our experiment, an optimally smoothed LOD series (Figure 1) derived from the Space 92 (Gross, 1993) combination of space-geodetic observations was used to estimate the LOD spectrum. Since the uncertainty of the LOI) series drops below 0.04 msec after 1985, only the data from 1985 to 1992 was used to estimate the spectrum.

The effects of solid earth tides (Yoder et al., 1981) and long period (fornightly and monthly) ocean tides (Dickman, 1993) were removed from the time series. We also removed from the data the annual and semi-annual variations by using the following seasonal model:

$$LOD_{seasonal} = 0.2880cos(2\pi t) + 0.1704sin(2\pi t) - 0.1433\, cos(4\pi t) - 0.2569sin(4\pi t), \quad (21)$$

where $LOD_{seasonal}$ is in msec and $t = (JD - 2447527.5)/365.2422$. This model was derived from approximately 30 years of LOD data which was provided by Gross (private communication, 1993). Since we are interested in LOD variability at periods much less than one year, a one year moving-average was removed from the series. The approximate spectrum of the remaining series was computed using the spectral analysis procedure described by Eubanks et al. (1985) with a 15 point spectral smoothing (Figure 2).

Based on the estimated spectrum (Figure 2), two stochastic models for the LOD were constructed. The first, which will be denoted as Model 1, is the sum of a random walk process (RW) and a first order auto-regressive process (AR1). This corresponds to the general model of equation 9, with $m = 2$. That is,

$$L(t) = Y_1(t) + Y_2(t), \quad (22)$$

where the RW process $Y_1(t)$ satisfies the equation

$$\dot{Y}_1(t) = W_1(t),\ q_1 = 3.0 \times 10^{-4}\, msec^2/day^3, \quad (23)$$

and the AR1 process $Y_2(t)$ satisfies the equation

$$\dot{Y}_2(t) + 0.077\, Y_2(t) = W_2(t),\ q_2 = 3.(I \times 10^{-3}\, msec^2/day^3. \quad (24)$$

For $i = 1,2$, q_i is the excitation power spectral density of the white noise process $W_i(t)$. For the AR1 process in equation (24), the constant 0,077 corresponds to a dissipation time of 1.3 days. The power spectrum corresponding to the process $L(t)$ given by equations (24)-(26) is shown in Figure 2.

The second model, which is denoted as Model 2, is the sum of Model 1 plus a second order auto-regressive process (AR2), which attempts to model the 40-50 day oscillation exhibited by the LOD spectrum, The model is given by

$$L(t) = Y_1(t) + Y_2(t) + Y_3(t), \quad (25)$$

where $Y_1(t)$ and $Y_2(t)$ are as in equations (24) and (25), but with $q_1 =$ S.0 x 10–i $msec^2/day^3$ and q_2 = 9.0 x 10^{-4} $msec^2/day^3$. The process $Y_3(t)$ satisfies the equation

$$\ddot{Y}_3(t) + 0.133\,\dot{Y}_3(t) + 0.0366 Y_3(t) = W_3(t), q_3 = 2.25 \times 10-' \; msec^2/days^3, \qquad (26)$$

which corresponds to a dissipation time of 15 days and a resonance period of 35 days. The power spectrum of the process $L(t)$ as given in equation (27) -(2S) is shown in Figure 2.

The current JPL Kalman Earth Orientation filter (KEOF) uses the following random walk model for the LOD (Morobito et, al., 1987):

$$\dot{L}(t) = W_L(t), q_L = 0.0036 \; msec^2/day^3. \qquad (27)$$

This is a special case of equation (6) with m = O. This model will be refered to as Model 0. The corresponding power spectrum is also shown in Figure 2.

Using the KEOF and IRIS (International Radio Interferometric Surveying) Intensive single-baseline VLBI data, the above models for LOD were compared based on their ability to predict LOD in the presence of long data gaps. The IRIS Intensive VLBI dataset spanned three years from October 1989 to the end of 1992, The sub-daily tidal effects were removed using the model of Herring (1994), The effects of the solid earth (Yoder et al., 1981) and ocean tides (Dickman, 1993) were also removed, as were the seasonal variations based on the model in equation (23). Systematic differences in terms of relative bias and rates between the IRIS Intensive series and the UT 1 series of Space 92 were also removed. In order to test the above models, forty gaps were inserted in the dataset, each of length thirty days. Using each model separately, the KEOF forward filter series is compared with the Space 92 series as described in this section. In the absence of data, the KEOF forward filter series is equivalent to the propagation of initial data thirty days into the future using the dynamical models described in section 2. The ability of the model to predict LOD is measured by the root mean square(RMS) of the difference between the forward filter series during the data gaps and the Space 92 series. The RMS of the prediction error for day i is

$$RMS(i) = \sqrt{\frac{1}{40}\sum_{n=1}^{40}\left(pred(i,n) - Space92(i)\right)^2} \;, 1 \leq i \leq 30 \;, \qquad (28)$$

where $pred(i,n)$ is the prediction of LOI) for the i^{th} day of the n^{th} gap, $1 \leq n \leq 40$,

In order for the prediction error to be reinitialized before entering each data gap, two successive gaps are separated by ninty days of daily data points. Consequently, the difference between the KEOF forward filter run and Space 92 decreases to a small level prior to entering any of the data gaps, Moreover, this initial error is approximately the same for each of the LOD models used.

4 Results

The RMS function for each LOD model is plotted against time (30 days) in Figure 3. The corresponding RMS function for the UT1 is plotted in Figure 4. Compared with the LOD prediction error of the KEOF model, both Model 1 and Model 2 reduce the 1,01) prediction error by a factor of approximately 10(% (day 1) to 20% (day 8). The corresponding reduction of tile UT1 prediction error varies from 13% (day 1) to 23% (day 8). For the first 15 days in prediction, the reduction of the LOD prediction error in Model 2 is approximately 3% to 5% greater than that of Model 1. For the second 15 days of prediction, the situation is reversed with roughly the same variability, The reduction of the UT1 prediction error in Model 2 is approximately 0% to 8% greater than that in Model 1. I'bus, Model 2 performs as well or better than Model 1 during the first 15 days of LOD prediction, and during the first 22 days of UT1 prediction. This suggests that modeling the 40-50 day LOD oscillation improves LOD predictions up to 15 days, and improves UT1 predictions up to 22 days. Compared with the KEOF model (Model O), the major improvement in the predictions can be attributed to the AR1 process, which is present in both Model 1 and Model 2.

in Figures 5a, we have plotted the actual LOD prediction error (RMS function described above) for Model O along with the LOD error as predicted by the Kalman filter. A similar plot for the UT1 prediction error is shown in Figure 5b. The corresponding errors for Model 1 and Model 2 are shown in Figures 5c-5f. For the first 16 days, the actual and predicted LOD errors corresponding to Model O and Model 2 (Figures 5a and 5e) coincide to a great extent. For the remaining 14 days, the predicted error for Model O overestimates the actual error by as much as 0.1 msec., while the errors corresponding to Model 2 continue to track each other. The predicted error corresponding to Model 1 (Figure 5c) begins to underestimate the actual error after only 8 days of prediction. The largest difference is approximately 0.05 msec. (day 13), and the two errors coincide again after 24 days of prediction, The difference between the predicted and actual UT1 errors corresponding to each model show similar behavior (Figures 5b, 5d, and 5f).

5 Discussion

As stated in the previous section, the improvements in the LOD and UT] predictions as exhibited by Model 1 and Model 2 can be attributed primarily to the AR1 component of the LOD models, which is not present in Model 0. It also appears that the initial error growth obeys a $t^{1/2}$ law for the first 15 days ($t \leq 15$). This is the error growth law of a random walk, which is precisely the assumption of Model O (Figure 5a). This growth law is also reflected by Model 2 (Figure 5e) for at least the first 10 days. However, the actual error begins to decline after 18 days and then begins to grow again after 26 days (Figure 3). Model 2 (Figure 5e) anticipates this behavior in error growth early on and thus tracks the actual error more closely than the other two models. It might therefore be suggested that the change in the actual error growth is due to the 40-50 day oscillation which is accounted for in Model 2. On the other hand, what we refer to as the 'actual error' in this paper is the observed error. It is not clear how accurate the observed error is with respect to the theoretically true error. The observed error (Figure 3) is a statistical error based on 40 ensembles of 30 day predictions using Space 92 as a reference series. Other experiments (Freedman et al., 1994) revealed similar results for Model O, but we know of no other results for ARMA models such as Model 1 and Model 2.

For practical considerations, we regard the first 15 days of prediction as being more important than the second 15 days, as our predictions are currently done on a twice a week basis. AS a result, the parameters that were chosen for Model 1 and Model 2 were based on inspection of the estimated spectrum of the Space 92 series. In general, optimal estimation of ARMA parameters based on data can be achieved, for example, using the extended Kalman Filter. However, in addition to being more costly to implement, such optimal fitting of the data may not reflect our preference for the first 1,5 days of prediction,

In conclusion, the main result from our LOD modeling experiments is that LOD is dominated by an AR1 structure in addition to a random walk structure at long periods. Adding the 40-50 day oscillation to the AR1 and random walk only slightly improves LOD predictions for the first 15 days. However, accounting for the 40-50 day oscillation seems to yield the most accurate predictions of the formal errors corresponding to LOD and UT1 predictions, Based on the results of our experiments, Model 2, as described in this paper, seems to be the most appropriate model for 15 day predictions of LOD and UT1.

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6 Figure Captions

Figure 1: Space 92 LOD series with the seasonal model and a one year moving average removed. Daily points are plotted from 1991-1992 only, but the spectrum in Figure 2 is computed from daily Space 92 points ranging from 1985-1992.

Figure 2: Estimated power spectrum of the Space 92 LOD series (using the procedure in Eubanks et al., (1 985) with a 15 point spectral smoothing) along with the theoretical power spectral densities of Model O, Model 1, and Model 2. Note: 'The spectrum is computed from Space 92 data ranging from 1985-1992.

Figure 3: RMS of LOD prediction error corresponding to 3 models. The error is computed as (Space 92- LOD forward filter run).

Figure 4: RMS of UT1 prediction error corresponding to 3 models.

Figure 5: Predicted and actual Standard deviation of LOD and UT1 prediction error (as a function of time) corresponding to Model O, Model 1, and Model 2.

Figure 1 - Space 92 LOD series with seasonal model and one year moving average removed

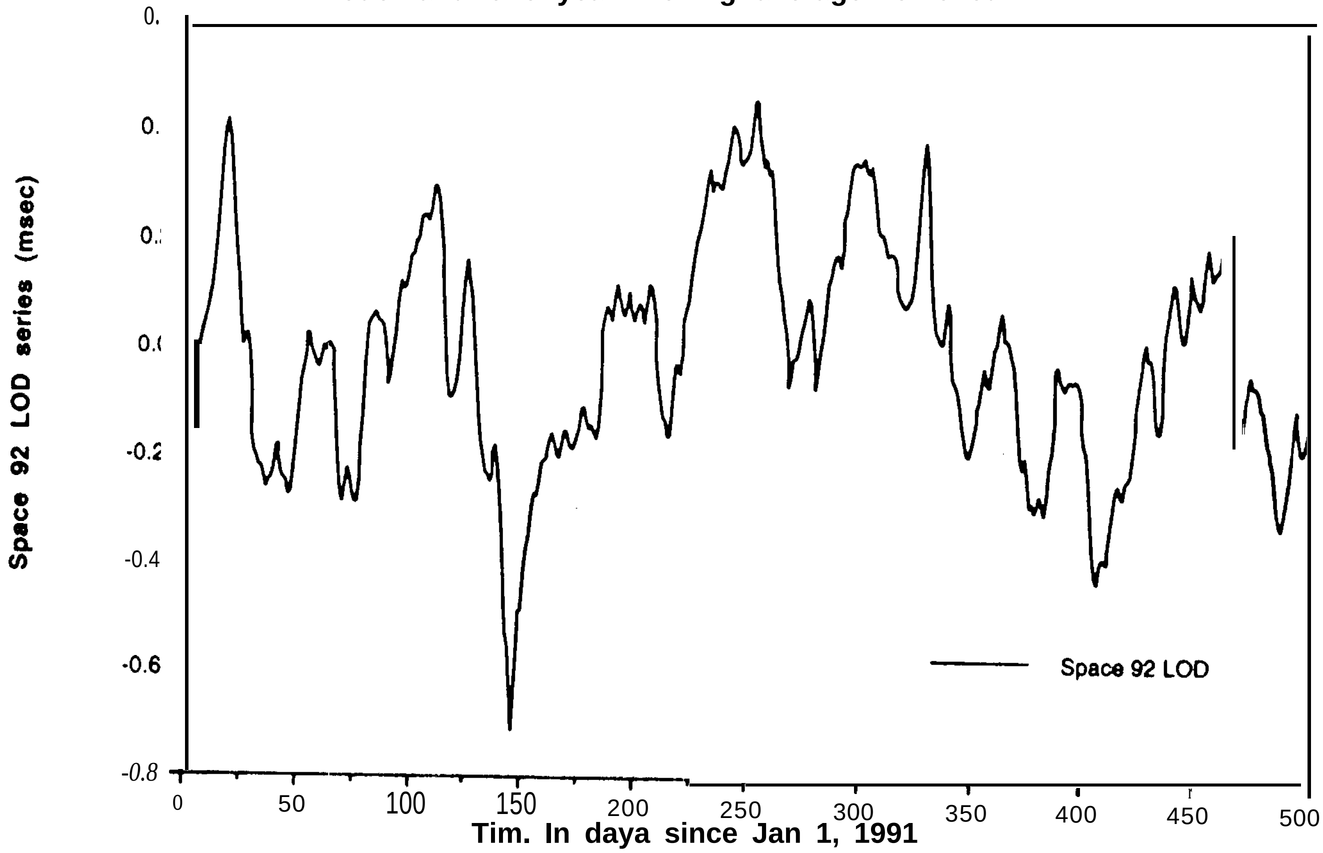


Figure 2 - Power Spectrum of Space 92
LOO series and LOD ARMA models

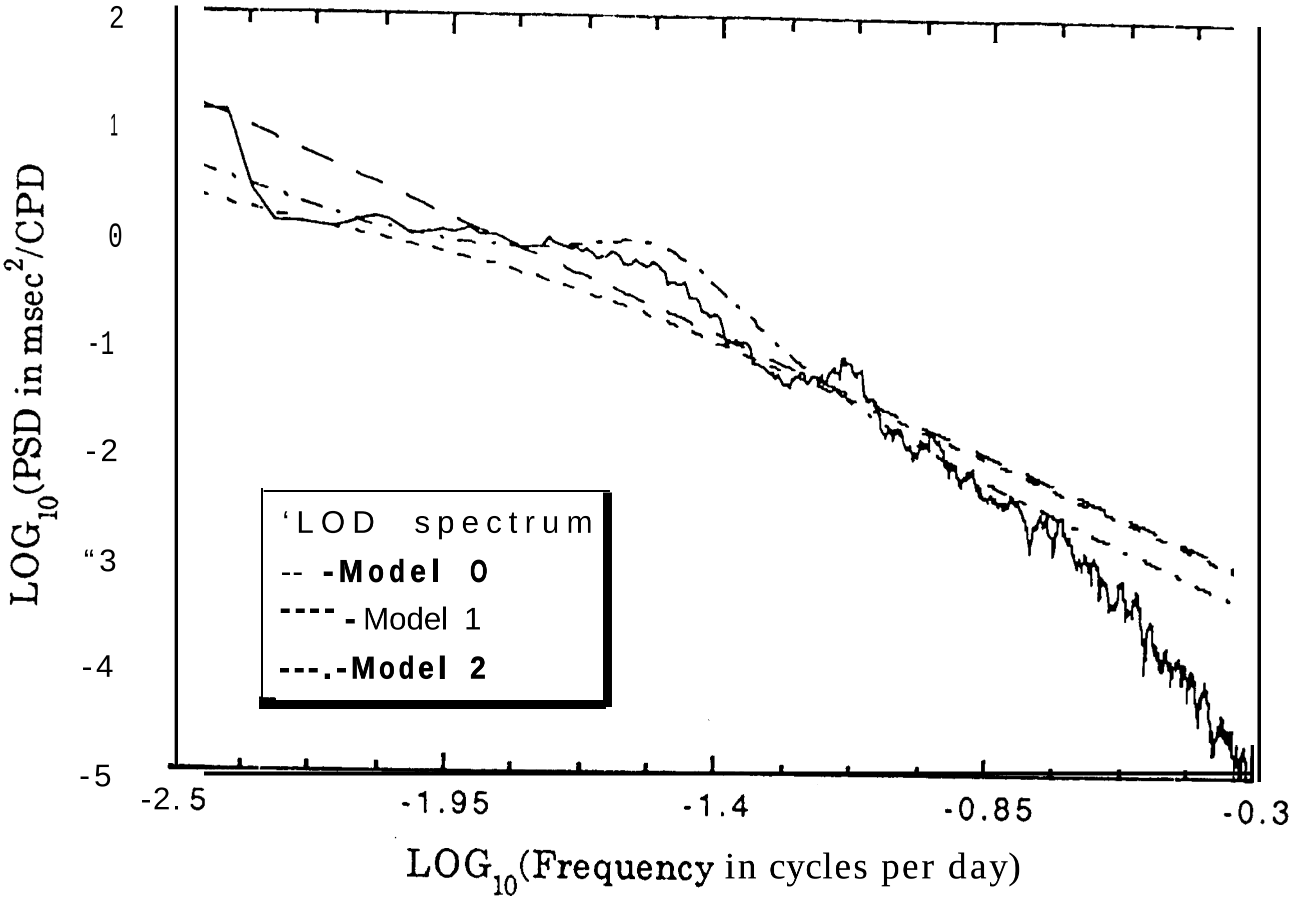


Figure 3 - LOD Prediction error RMS vs. day number

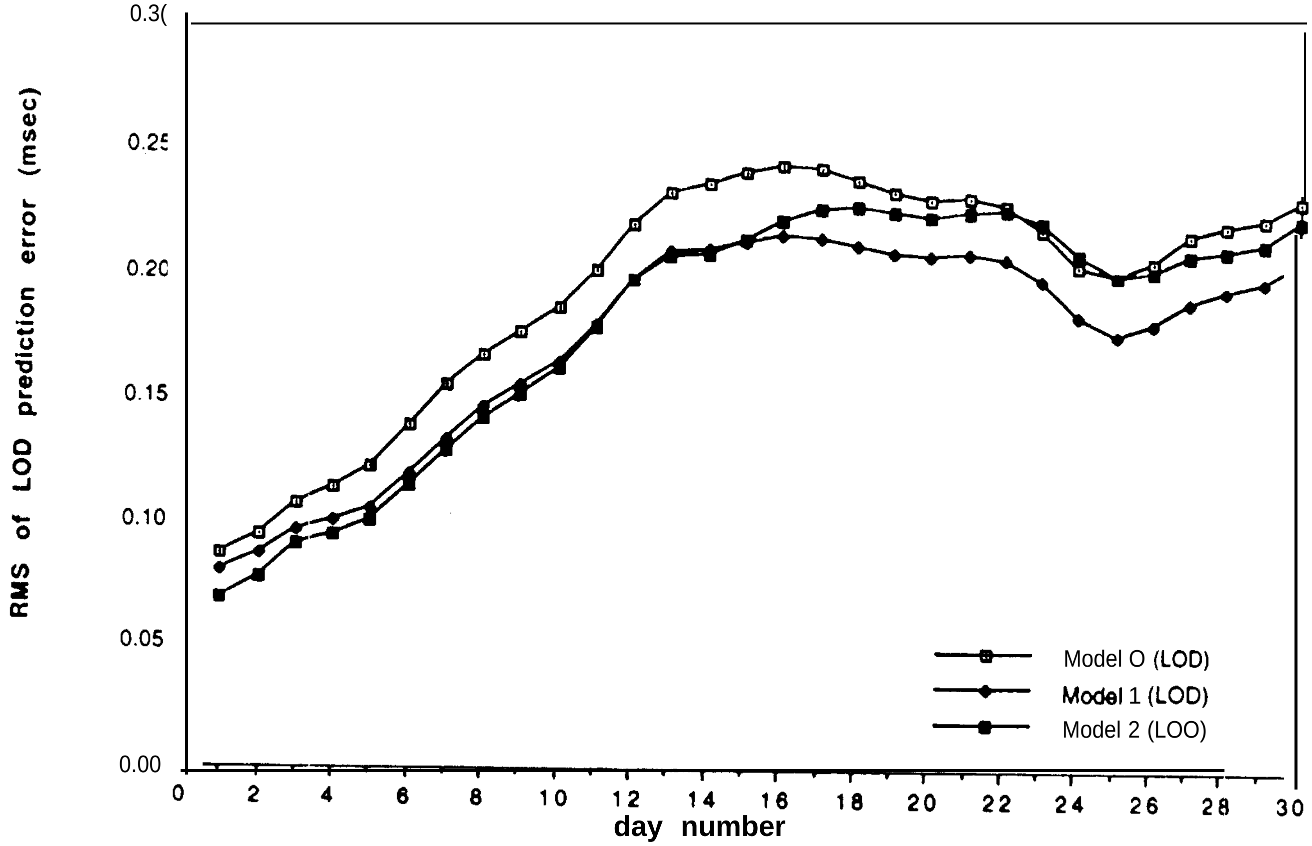


Figure 4 - UT1 Prediction error RMS vs. day number

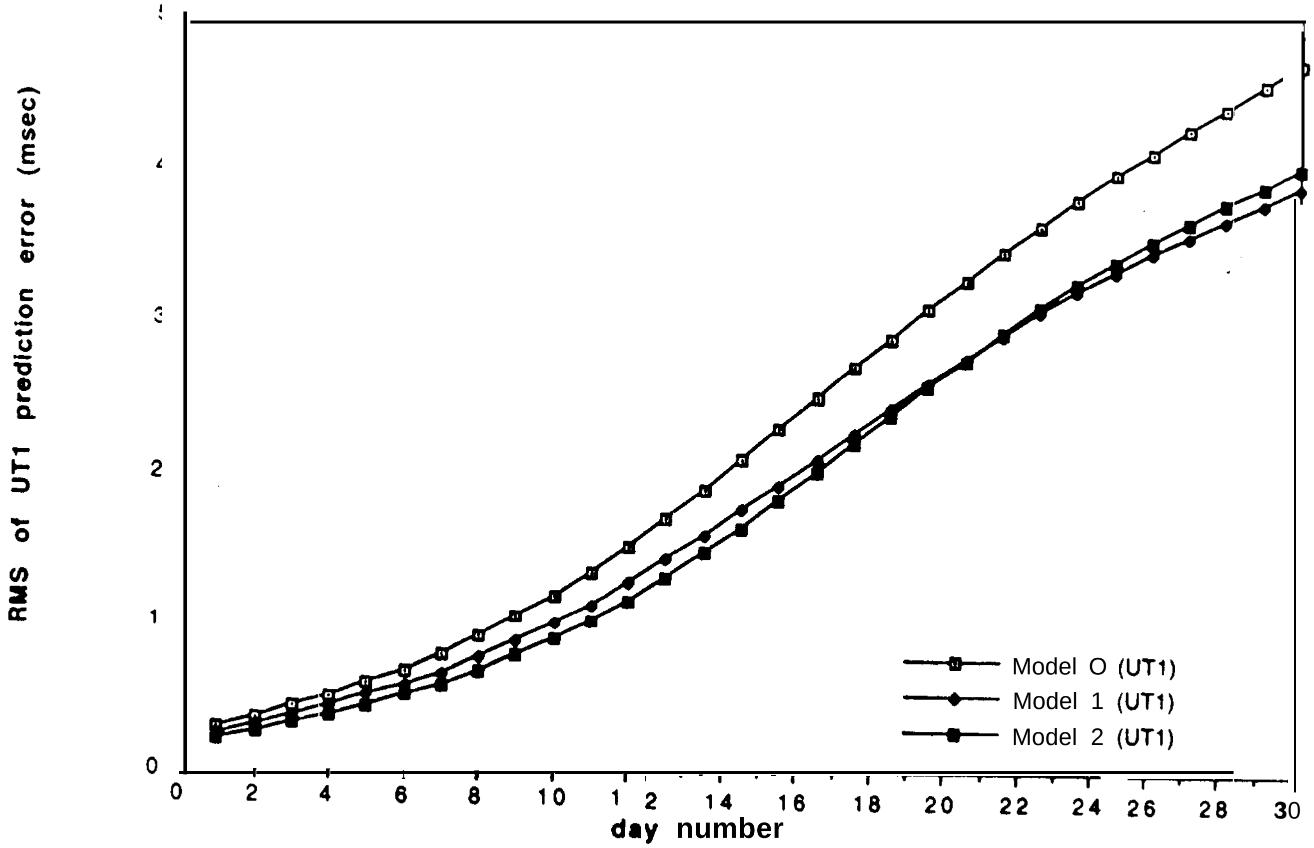


Figure 5a - Actual and estimated standard deviation of LOD Prediction error (Model O)

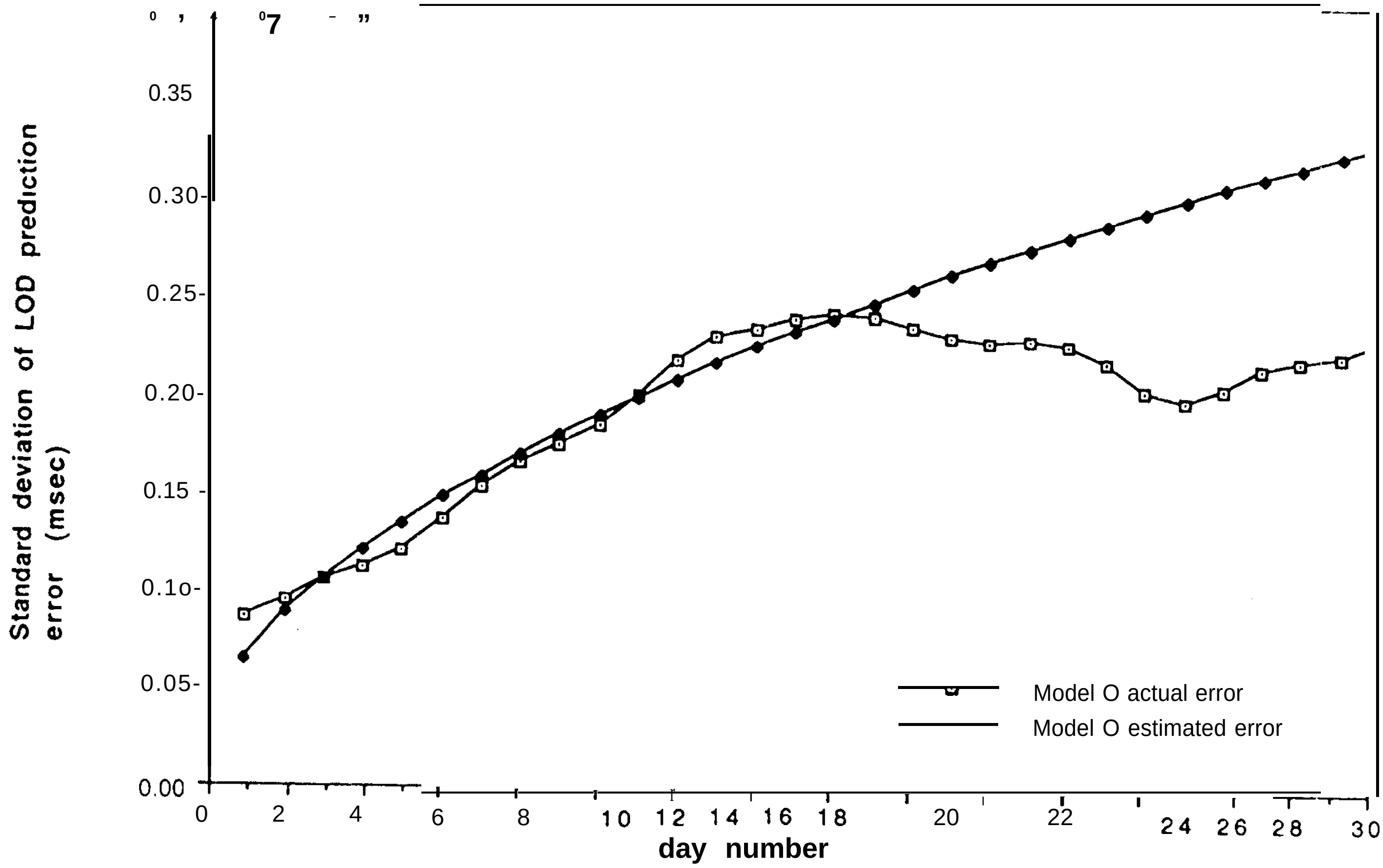


Figure 5b - Actual and estimated standard deviation of UTI Prediction error (Model O)

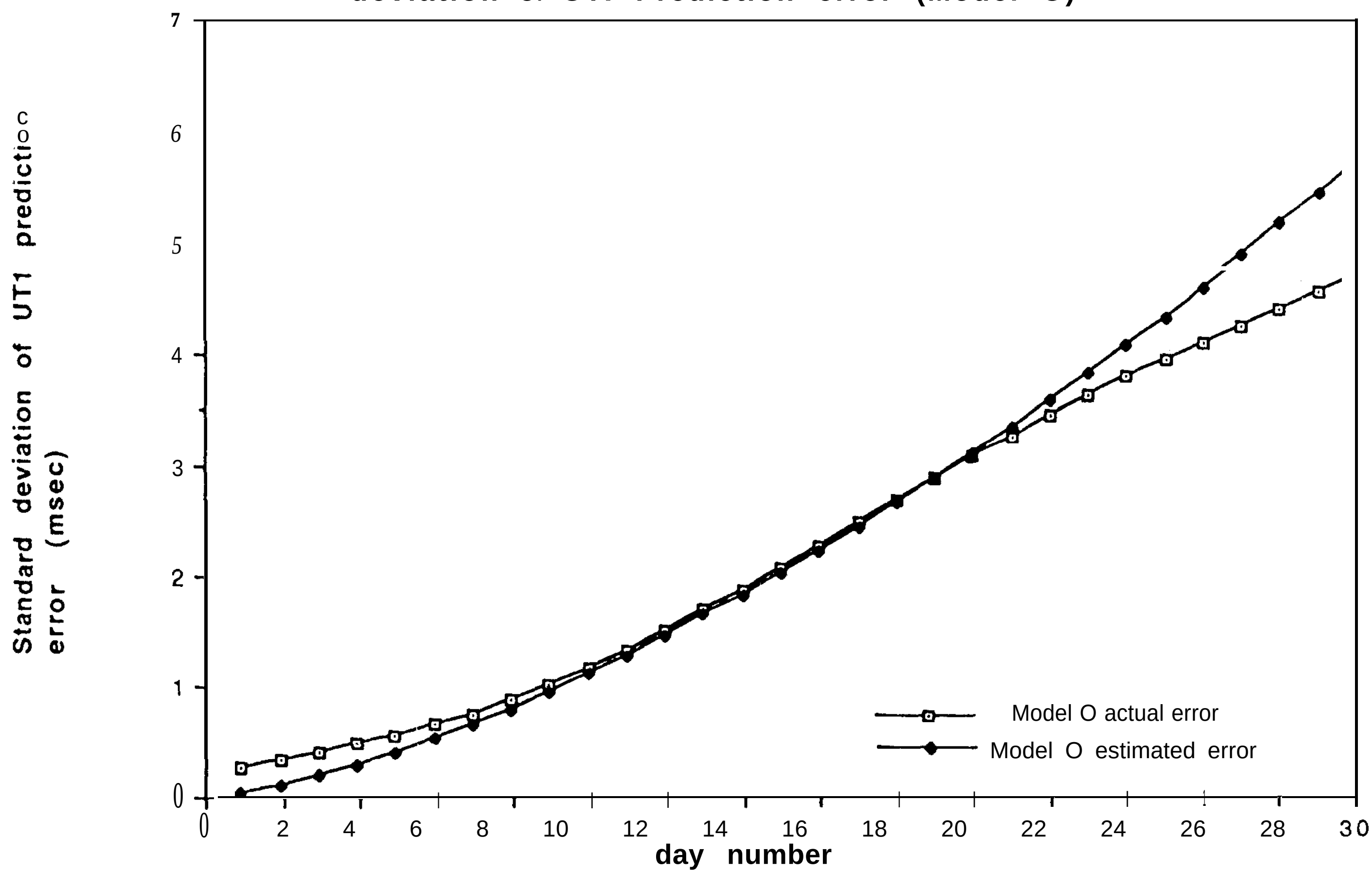


Figure 5c - Actual and estimated standard deviation of LOD Prediction error (Model 1)

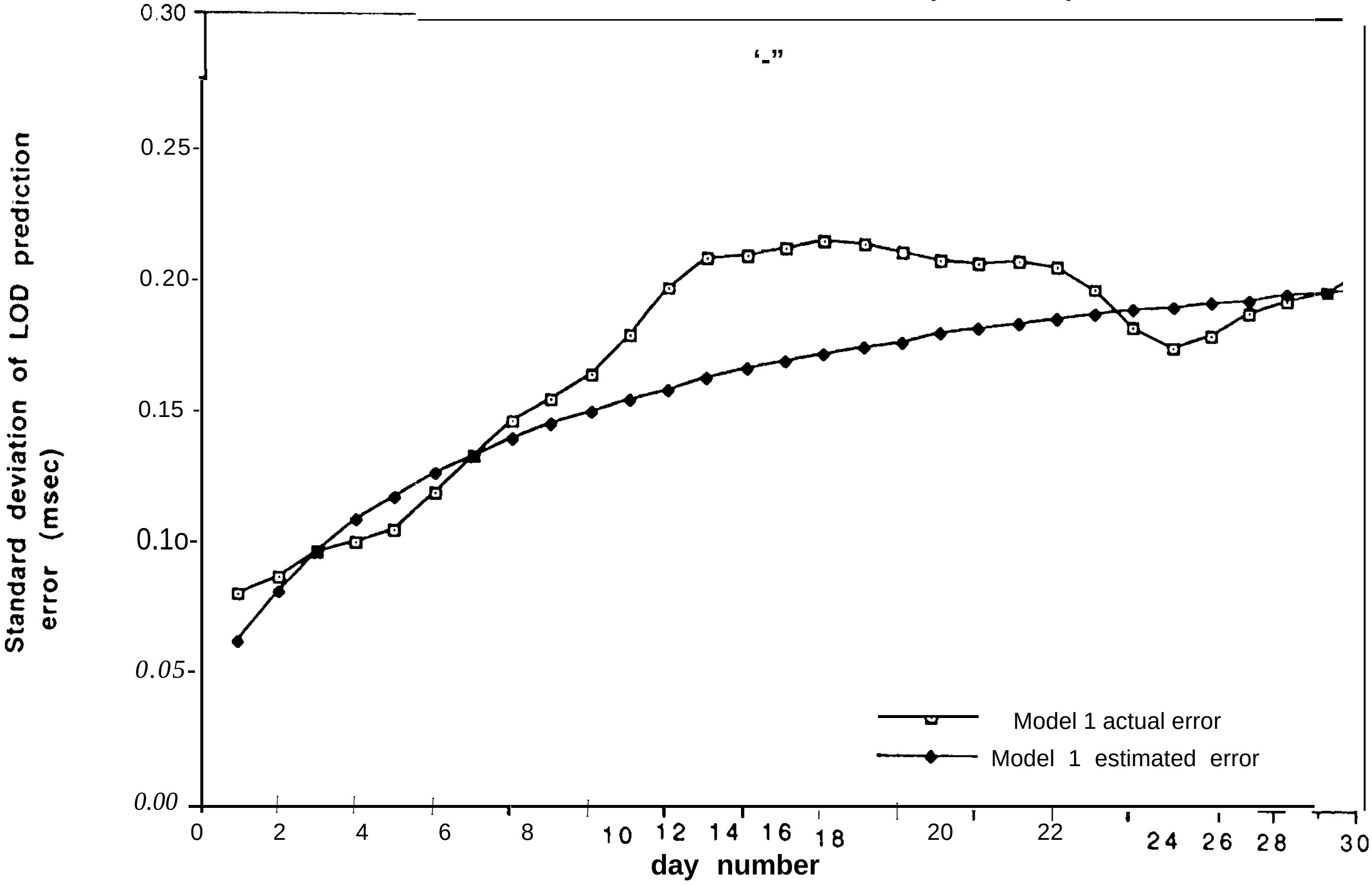


Figure 5d - Actual and estimated standard deviation of UT1 Prediction error (Model 1)

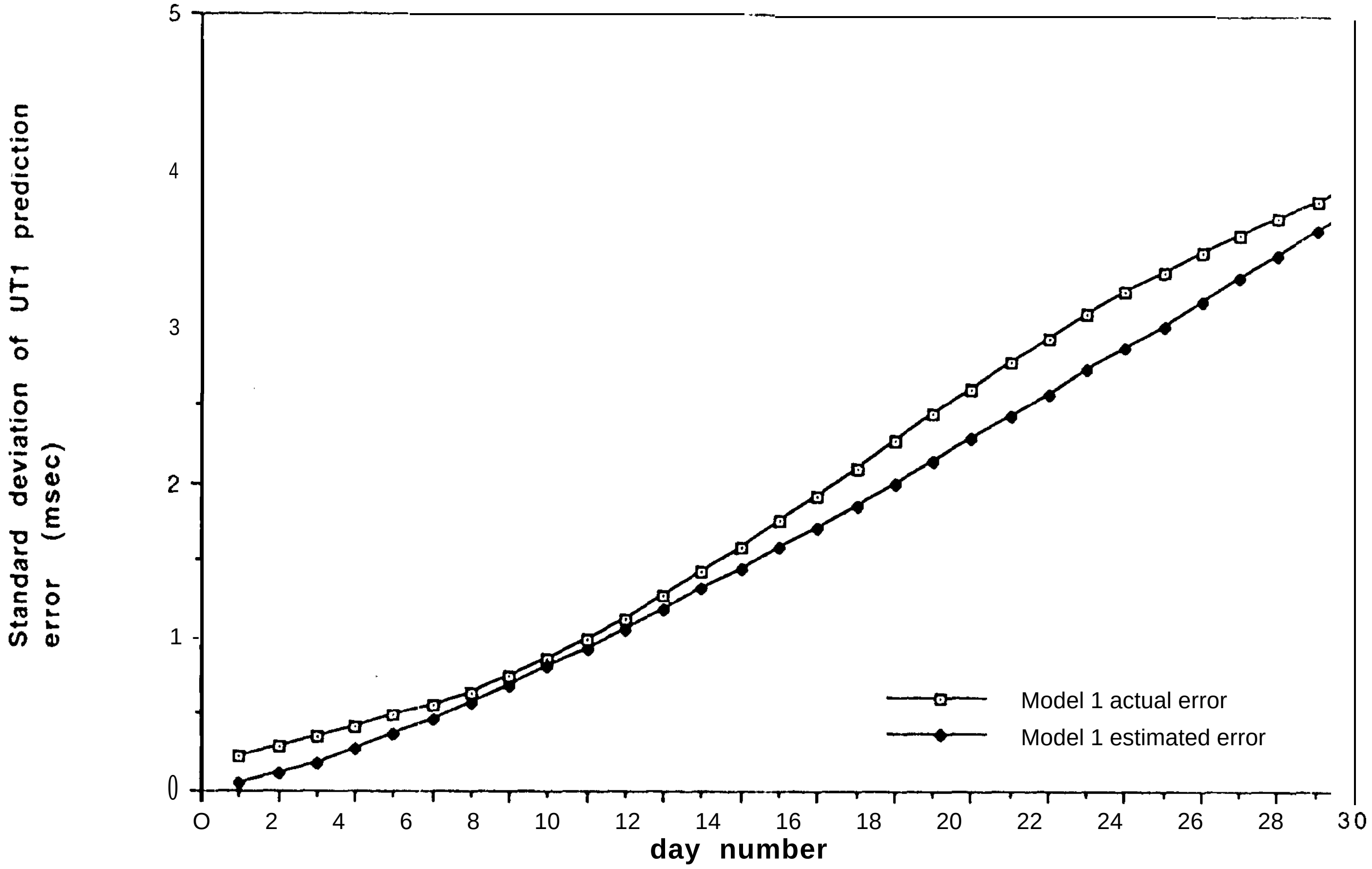


Figure 5e - Actual and estimated standard deviation of LOD Prediction error (Model 2)

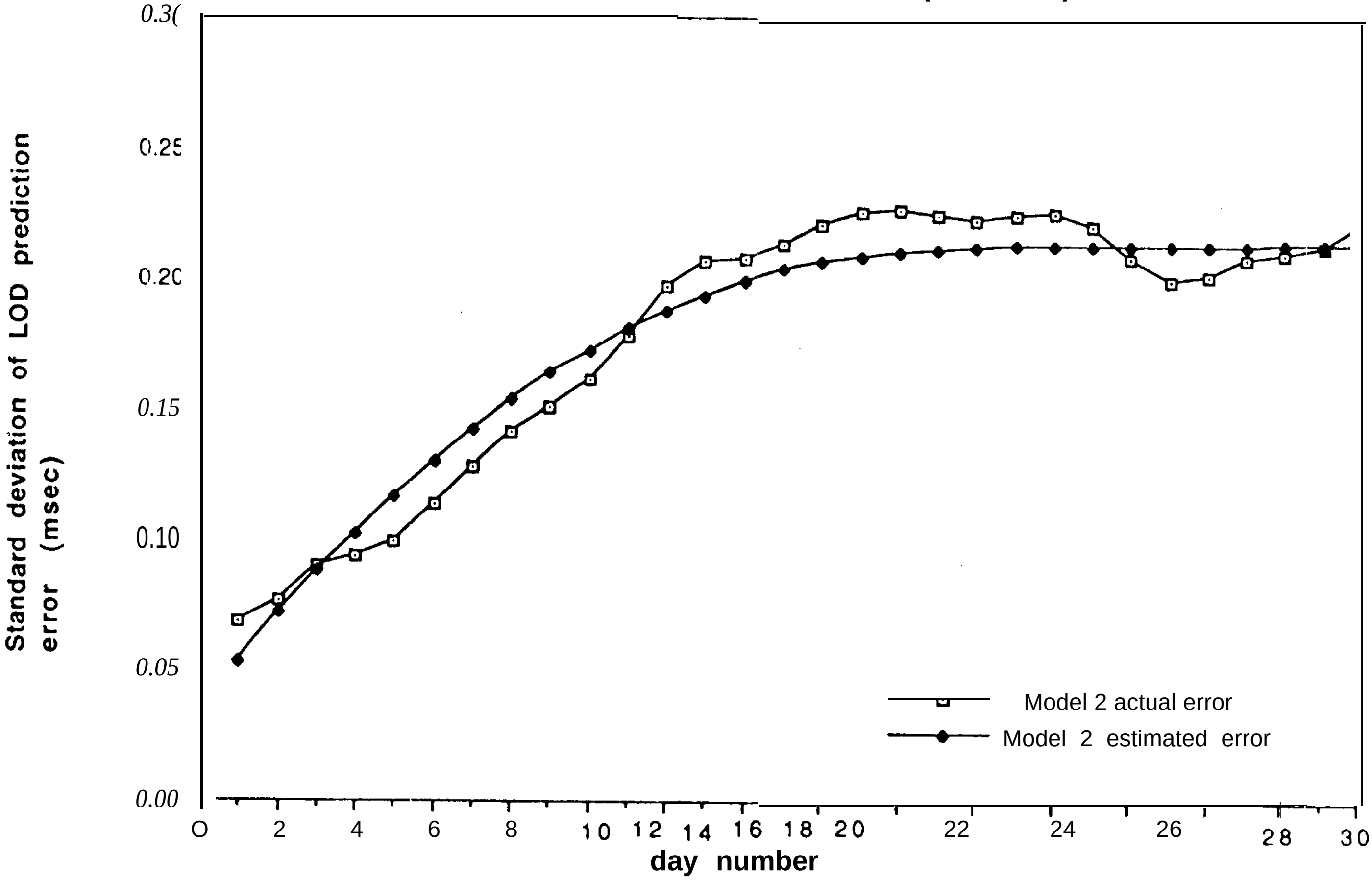


Figure 5f - Actual and estimated standard deviation of UT1 Prediction error (Model 2)

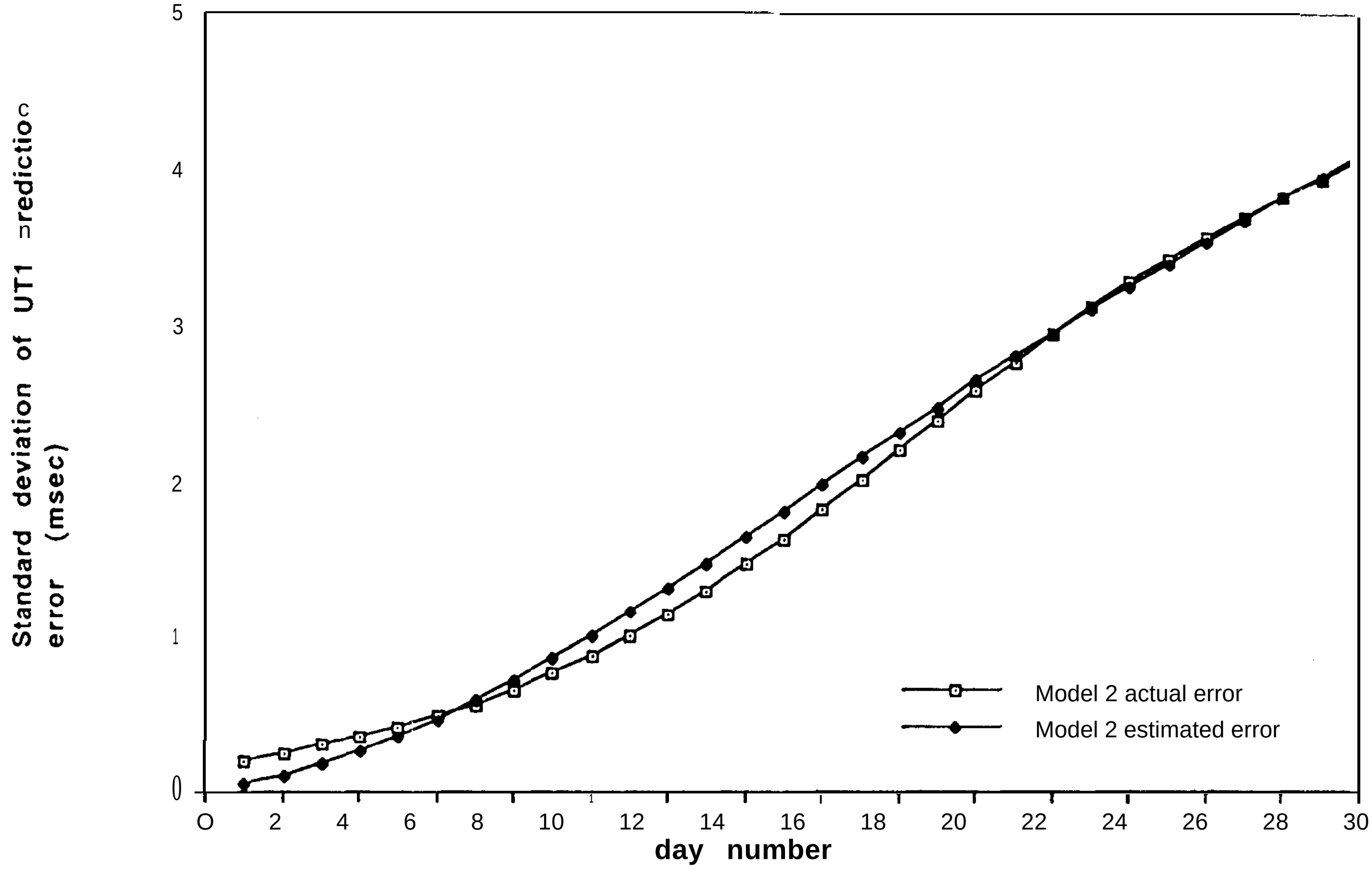


Figure 5a. Actual and estimated standard deviation of LOD Prediction error (Model 0)

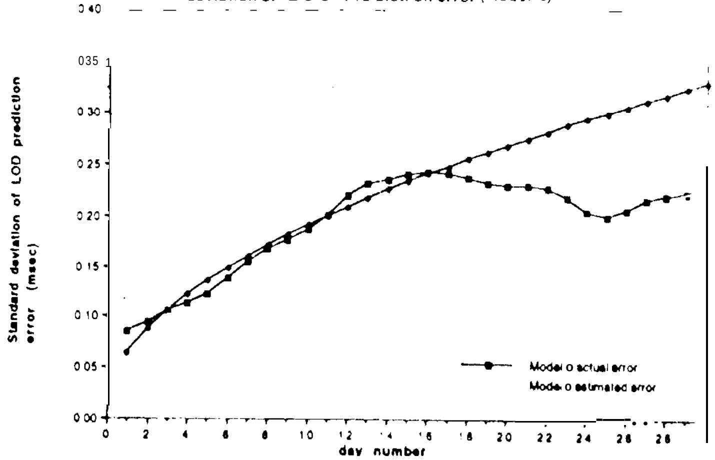


Figure 5b Actual and estimated standard deviation of UT1 Prediction error (Model 0)

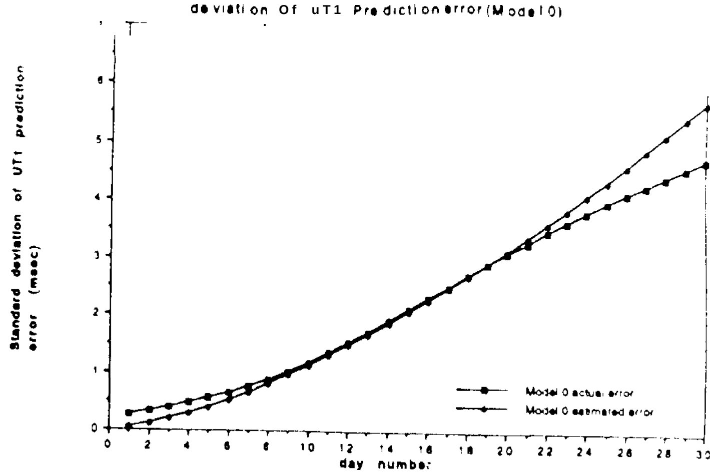


Figure 5c . Actual and estimated standard deviation of LOD Prediction error (Model 1)

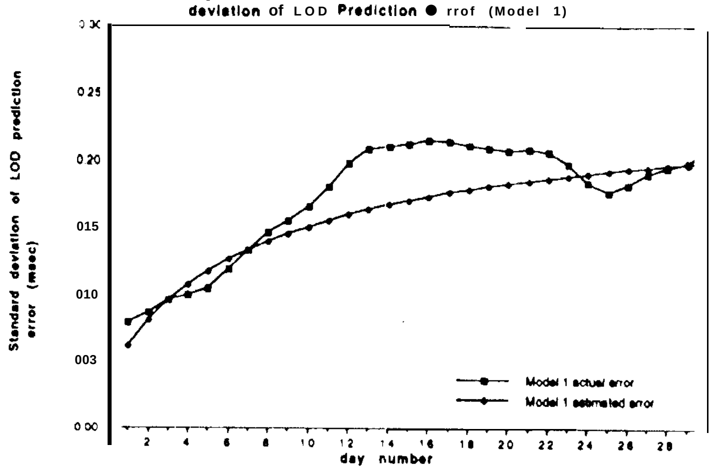


Figure 5d . Actual and estimated standard deviation of UT1 Prediction error (Model 1)

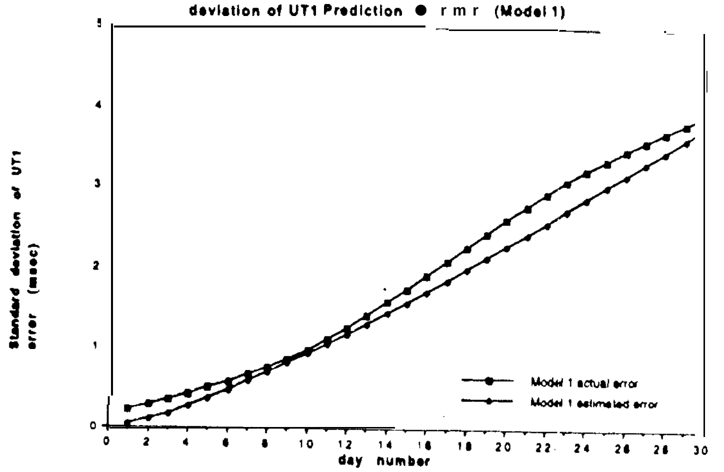


Figure 5e . Actual and estimated standard deviation of LOD Prediction error (Model 2)

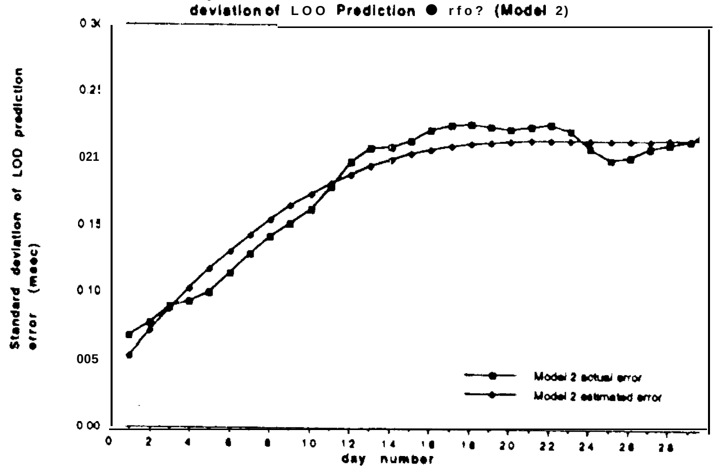
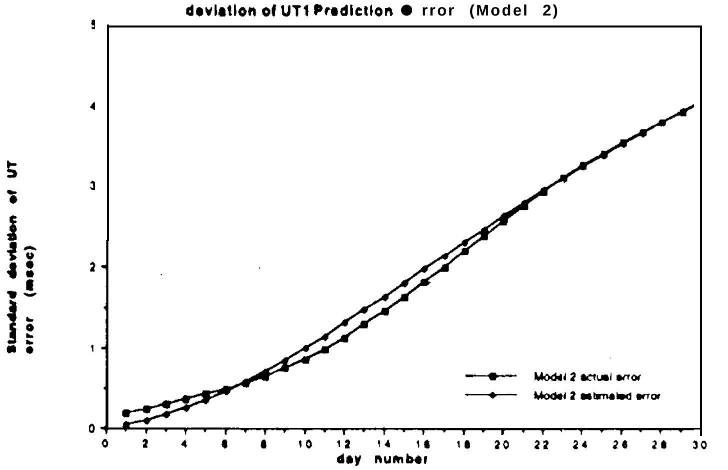


Figure 5f . Actual and estimated standard deviation of UT1 Prediction error (Model 2)



Appendix A: FORMULATION OF THE RANDOM EARTH ROTATION MODEL

The equation relating the UT1 with the LOD is given by

$$\dot{U}(t) = -L(t) = -\frac{\Delta\Lambda(t)}{\Lambda_0} , \tag{1}$$

where $U(t)$ denotes UT1, $\Delta\Lambda(t)$ denotes the variability in LOD, and $\Lambda_0 = 86400$ sees. The LOD is assumed to satisfy the differential equation

$$\dot{L}(t) = Y_1(t) + Y_2(t) + \ldots + Y_m(t) + W_L(t) , \tag{2}$$

where each of the Y_i's belongs to the set ARMA(n, n-1) and W_L is white noise. As in Section 2, since each Y_i belongs to ARMA(n,n-1), the nth-order differential equation that is satisfied by Y_i can be reduced to the first-order system

$$\dot{X}(t) = A_i X_i(t) + B_i W_i(t) \qquad X_i(t) = (X_1(t) \; X_2(t) \ldots X_n(t))^T \tag{3}$$

$$Y_i = C_i X_i \tag{4}$$

where A_i, B_i, C_i are of the form given in equation (4) of Section 2. Equation (4) can be substituted into equation (2), and the resulting equation, combined with equations (1) and (3), can be written in state-space form as follows:

$$\begin{bmatrix} \dot{U} \\ \dot{L} \\ \dot{X}_1 \\ \dot{X}_2 \\ \vdots \\ \dot{X}_m \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & \cdots & & 0 \\ 0 & 0 & C_1 & C_2 & \cdots & C_m \\ 0 & 0 & A_1 & 0 \ldots & 0 & \\ \vdots & & & A_2 & & \\ \vdots & & & & & \\ 0 & \cdots & & & & A_m \end{bmatrix} \begin{bmatrix} U \\ L \\ X_1 \\ X_2 \\ \vdots \\ X_m \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & & & \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & B_1 & 0 & \cdots & 0 \\ & & & B_2 & & \\ 0 & & & & & B_m \end{bmatrix} \begin{bmatrix} 0 \\ W_L \\ W_1 \\ W_2 \\ \vdots \\ W_m \end{bmatrix} \tag{5}$$

The system (5) can be abbreviated as

$$\dot{\mathcal{Y}} = F\mathcal{Y} + \mathcal{G}\mathcal{W} , \tag{6}$$

where F and G are $(mn + 2) \times (mn + 2)$, and $\mathcal{Y}$, $\mathcal{W}$ are $(mn + 2) \times 1$ column vectors. The matrix F can be partitioned as $F = F_1 + F_2$ where

$$F_1 = \begin{bmatrix} 0 & -1 & 0 & \cdots \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \\ \vdots & & & \\ 0 & \cdots & \cdots & 0 \end{bmatrix} , \quad F_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \ldots\ldots 0 \\ 0 & 0 & C_1 & C_2 \ldots C_m \\ & & A_1 & \\ & & & A_2 \\ & & & & A_m \end{bmatrix}$$

The following properties will be used:

$P1/\; F_1^n = 0 \qquad$ for $n \geq 2$.

$P2/\; F_2^m F_1^n = 0 \qquad$ for $n \geq 1, m \geq 1$.

It will be shown in P3 that F_2^m has nonzero entries in the same location as F_2. Also, $F_1^n = 0$, so we need only show that $F_2 F_1 = 0$. This is easily seen by direct computation.

P3/ For $m \geq 2$, F_2^m has nonzero entries in the same location as F_2. A formula for F_2^m will be useful in the analysis to come, and it will show that P3 holds for $m \geq 2$. We first assume that A_i is invertible.

$$F_2^m = \begin{bmatrix} 0 & 0 & \cdots & & & 0 \\ 0 & 0 & P_1^{m-1} & \cdots & & P_m^{m-1} \\ & & A_1^m & & & \\ & & & A_2^m & & \\ & \ddots & & & & \\ & & & & & A_m^m \end{bmatrix},$$

where the row vector $P_i^m = C_i A_i^m$ for $m \geq 0$.

The above formula is easily proven by induction.

$P4/\ (F_1 + F_2)^n = F_1 F_2^{n-1} + F_2^n \qquad n \geq 1.$

Proof (by induction):

$$(F_1 + F_2)(F_1 + F_2) = F_1^2 + F_1 F_2 + F_2 F_1 + F_2^2$$
$$= 0 + F_1 F_2 + 0 + F_2^2 = F_1 F_2 + F_2^2.$$

Assume

$(F_1 + F_2)^{n-1} = F_1 + F_2^{n-2} + F_2^{n-1}$. Then

$$(F_1 + F_2)(F_1 + F_2)^{n-1} = (F_1 + F_2)\left(F_1 F_2^{n-2} + F_2^{n-1}\right)$$
$$= F_1^2 F_2^{n-2} + F_1 F_2^{n-1} + F_2 F_1 F_2^{n-2} + F_2^n.$$

Using P1 and P2, P4 follows directly.

Using P1 through P4, the transition matrix $\Phi(t) = \exp\{(F_1 + F_2)t\}$ can now be derived,

$$\exp\{F_1 + F_2\}t = \sum_{n=0}^{\infty} \frac{t^n}{n!}(F_1 + F_2)^n \tag{7}$$

$$= I + (F_1 + F_2)t + \sum_{n=2}^{\infty} \frac{t^n}{n!}(F_1 + F_2)^n$$

$$= I + (F_1 + F_2)t + \sum_{n=2}^{\infty} \frac{t^n}{n!}\left(F_1 F_2^{n-1}\right) + \sum_{n=2}^{\infty} \frac{t^n}{n!} F_2^n \ . \tag{8}$$

Since the summations start at $n = 2$, we can use the formula in P3 to compute F^{n-1}. The expression for $F_1 F^{n-1}$ for $n \geq 2$ is given by

$$F_1 F_2^{n-1} = \begin{bmatrix} 0 & 0 & -C_1 A_1^{n-2} & \cdots & -C_m A_m^{n-2} \\ 0 & 0 & \cdots & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}.$$

After summing up from n = 2 to ∞, the nonzero elements of $F_1 F_2^{n-1}$ will be of the form

$$-C_i \sum_{n=2}^{\infty} \frac{t^n}{n!} A_i^{n-2} = C_i A_i^{-2} \left[I + A_i t - e^{A_i t} \right] \tag{9}$$

Equation (7) can now be rewritten as

$$\exp\{(F_1 + F_2)t\} = I + F_1 t + M_1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} F_2^n \tag{lo}$$

where M_1 can be computed using (9). Using the formula in P3 and bringing the summation inside the matrix yields

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} F_2^n = \begin{bmatrix} 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & R_1 & R_2 \cdots & \cdots R_m \\ & & e^{A_1 t} - I & & \\ & & & e^{A_2 t} - I & \\ & & & & \ddots \\ & & & & e^{A_m t} - I \end{bmatrix},$$

where the row vector $R_i = C_i A_i^{-1}[e^{tA_i} - I]$ Finally, (10) becomes

$$e^{(F_1+F_2)t} = \begin{bmatrix} 1 & -t & Z_1 & Z_2 & \cdots & Z_m \\ 0 & 1 & R_1 & R_2 & \cdots & R_m \\ & & e^{A_1 t} & & & \\ & & & e^{A_2 t} & & \\ & & & & \ddots & \\ & & & & & e^{A_m t} \end{bmatrix} \tag{11},$$

where $Z_i = C_i A_i^{-2}[I + t A_i - e^{tA_i}]$, and $R_i = C_i A_i^{-1}[e^{tA_i} - I]$

We now consider the second case, in which the coefficients of A_i satisfy a_i = O, for i = O, 1, ... $n-1$, and b j = O f O r j = 1, 2, . . . , n - 1. The differential equation becomes $\frac{d^n y}{dt^n}$ = bow(t), and the matrix A_i becomes

$$A_i = \begin{bmatrix} 0 & 1 & 0 \cdots\cdots & & \\ 0 & 0 & 1 & 0 \cdots\cdots & \\ \vdots & & & & \\ 0 & 0 & \cdots\cdots & 0 & 1 \\ 0 & 0 & \cdots\cdots & 0 & 0 \end{bmatrix}, \quad C_i^T = \begin{bmatrix} b_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{12}$$

For n = 1, the model is a random walk; for n = 2, it is an integrated random walk. For this class of excitation models, an equation similar to (11) will be derived.

The matrix A of the form (12) haa the following property:

$$A^k = \{a_k(i,j)\} \text{ has entry 1 if } j = 1 + k \text{ and entry zero otherwise.} \tag{13}$$

Hence, if A is an $n \times n$ matrix, then $An = O$. Therefore, the matrix

$$\sum_{k=0}^{\infty} A^k \frac{t^k}{k!} = \sum_{k=0}^{n-1} A^k \frac{t^k}{k!} \tag{14}$$

is given by

$$\begin{vmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{3!} & \cdot & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{n-2}}{(n-2)!} \\ 0 & 0 & 1 & \cdot & \cdot & \\ & & & & & \\ \cdot & & & & & t \\ 0 & & & & & 1 \end{vmatrix} \tag{15}$$

and the matrix

$$\sum_{k=2}^{\infty} \frac{t^k}{k!} A^{k-2} = \sum_{k=2}^{n+1} \frac{t^k}{k!} A^{k-2}$$

is given by

$$\begin{vmatrix} \frac{t^2}{2} & \frac{t^3}{3!} & \cdots\cdots\cdots & \frac{t^{n+1}}{(n+1)!} \\ 0 & \frac{t^2}{2} & \cdots\cdots\cdots & \cdot \\ & & & t^3/3! \\ 0 & \cdots\cdots & \cdots\cdots & t^2/2 \end{vmatrix} \tag{16}$$

The quantity Z_i appearing in (11) is given by $-C_i \sum_{k=2}^{n+1} \frac{t^k}{k!} A^{k-2}$, where $C_i = (b_0, 0, \ldots, 0)$. It follows that Z_i is given by

$$Z_i = -b_0 \left(\frac{t^2}{2} \; \frac{t^3}{3!} \cdots \frac{t^{n+1}}{(n+1)!} \right) \tag{17}$$

The expression R_i appearing in (11) is given by

$$R_i = C_i \sum_{k=1}^{\infty} \frac{t^k}{k!} A^{k-1} = C_i \sum_{k=1}^{n} \frac{t^k}{k!} A^{k-1}. \tag{18}$$

By a similar argument as above, it can be shown that

$$R_i = b_0 \left(t \; \frac{t^2}{2} \cdots \frac{t^n}{n!} \right) \tag{19}$$

Appendix B: Computation of Process Noise Covariance Matrix

In this section we will derive an expression for the process noise covariance matrix Q_k, which is given by (see equation 13 of section 2)

$$Q_k = \int_{t_k}^{t_{k+1}} \Phi(t_{k+1} - \tau) G Q_W G^T \Phi^T (t_{k+1} - \tau) \, d\tau. \tag{20}$$

In the process of evaluating Q_k, certain expressions appear repeatedly. One such expression is $e^{At} Q e^{A^T t}$, where Q is an $n \times n$ matrix that has entry zero everywhere except $Q(n, n) = q$ (*i.e.*, element (n, n) has nonzero entry q). Let the eigenvalues of A be numbered $\{\lambda_1, \lambda_2, \ldots \lambda_n\}$. Some of the eigenvalues may be repeated, in which case they should be numbered successively. Denote the n^{th} column of e^{At} by V_n. Then, by direct calculation, it can be shown that

$$e^{At} Q e^{A^T t} = q V_n V_n^T, \tag{21}$$

which is an n x n matrix

It is known that for .4 of the form given by equation (15) of section 2, e^{At} = V(t) V-l(0), where $V(0)$ contained eigenvectors or generalized eigenvectors, depending on the distinctness of the eigenvalues of matrix A(Faddeeva, 1959). The following analysis assumes that the eigenvalues are distinct. The general case of eigenvalues with multiplicities was treated by Hamdan (1993).

Let column n of matrix $V^{-1}(O)$ be given by $\eta_n = \{\xi_1, \xi_2, \ldots, \xi_n\}^T$. If $V_n = \{\mu_1, \mu_2, \ldots, \mu_n\}$, then the i^{th} entry of V_n, $1 \leq i \leq n$, is given by

$$\mu_i = \sum_{j=1}^{n} \xi_j \left(\lambda_j^{i-1} e^{\lambda_j t}\right). \tag{22}$$

It follows that element (i, t') of matrix $e^{At}Qe^{At}$ is given by (note: $V(1,j) = e^{\lambda_j t}, V(1,s) = e^{\lambda_s t}$)

$$q\mu_i\mu_\ell = q\sum_{j=1}^{n}\sum_{s=1}^{n} \xi_j \xi_s \lambda_j^{i-1} \lambda_s^{\ell-1} e^{(\lambda_j+\lambda_s)t} \tag{23}$$

Thus, element (i, ℓ) of matrix $\int_{t_k}^{t_{k+1}} e^{At}Qe^{A^Tt}dt$ is given by $q\int_{t_k}^{t_{k+1}} \mu_i\mu_\ell dt$, which is easily evaluated

By letting At $= t_{k+1} - t_k$, we may rewrite Q_k as

$$Q_k = \int_0^{\Delta t} \Phi(s)GQ_WG^T\Phi^T(s)ds, \tag{24}$$

where Q_w is assumed to be of the form diagonal (O, $q_L, Q_{W_1}, \ldots, Q_{W_m}$), where q_L is a scalar and each $n \times n$ matrix Qw,, is of the form Q_{W_i} = diagonal (q_i, O, ... O). The matrix G is given by G = diagonal $(0, 1, B_1, \ldots, B_m)$, where the B_i's are as defined in equation (4), Section 2. It follows that the matrix Q = GQ_WG^T is given by

$$Q = \text{diagonal } (0, q_L, Q_1, \ldots, Q_m), \text{ where } Q_i = \text{diagonal } (0, 0, \ldots, q_i). \tag{25}$$

The transition matrix Φ will be partitioned as follows:

$$\Phi = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}, \text{where } M_1 = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}_{2\times 2}, \quad M_2 = \begin{bmatrix} Z_1 \ldots Z_m \\ R_1 \ldots R_m \end{bmatrix}_{2\times(mn)} \tag{26}$$

M_3 = (mn x 2) zero matrix, M_4 = diagonal $(e^{A_1 t}, \ldots, e^{A_m t})$ is mn x mn.

Let Q be partitioned as

$$Q = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix},$$

where

$$\alpha_1 = \begin{bmatrix} 0 & 0 \\ 0 & q_L \end{bmatrix}_{2\times 2}. \tag{27}$$

$\alpha_2 = 2 \times mn$ zero matrix, $\alpha_3 = (\alpha_2)^T$, and $\alpha_4 = \text{diagonal}(Q_1, .., Q_m)$ is $(mn \times mn)$ Using the above partitions, the matrix $\Phi Q \Phi^T$ will be given by

$$\Phi_T Q \Phi_T^T = \begin{bmatrix} \Psi_1(t) & \Psi_2(t) \\ \Psi_3(t) & \Psi_4(t) \end{bmatrix} , \tag{28}$$

where

$$\begin{aligned} \Psi_1(t) &= M_1\alpha_1 M_1^T + M_2\alpha_4 M_2^T \qquad ; 2 \times 2 . \\ \Psi_2(t) &= M_1\alpha_1 M_3^T + M_2\alpha_4 M_4^T = M_2\alpha_4 M_4^T \qquad ; 2 \times mn. \\ \Psi_3(t) &= M_4\alpha_4 M_2^T = \Psi_2^T(t) \\ \Psi_4(t) &= M_4\alpha_4 M_4^T \qquad ; mn \times mn. \end{aligned} \tag{29}$$

Each one of the above matrices will be integrated separately. $\Psi_4(t)$ is actually given by $\Psi_4(t)$ = diagonal $\left(e^{A_1 t} Q_1 e^{A_1^T t}, \ldots, e^{A_m t} Q_m e^{A_m^T t}\right)$. The integral of this matrix is of the form given in equation (27). $\Psi_2(t)$ is the transpose of $\Psi_3(t)$, so only $\Psi_1(t)$ needs to be analyzed.

The first term appearing in $\Psi_1(t)$ is given by

$$M_1\alpha_1 M_1^T = \begin{bmatrix} q_L t^2 & -q_L t \\ -q_L t & q_L \end{bmatrix} \tag{30}$$

where $q_L = E(W_L^2)$. The second term appearing in $\Psi_1(t)$ is given by

$$M_2\alpha_4 M_2^T = \sum_{i=1}^{m} \begin{bmatrix} Z_i Q_i Z_i^T & Z_i Q_i R_i^T \\ R_i Q_i Z_i^T & R_i Q_i R_i^T \end{bmatrix} . \tag{31}$$

Let $E(W_i^2) = q$, and recall from Equation (16)-(17) of Section 2 that

$$\begin{aligned} Z_i &= C_i A_i^{-2} (I + A_i t - e^{A_i t}) \\ R_i &= C_i A_i^{-1} (e^{A_i t} - I) \end{aligned} \tag{32}$$

whenever A_i^{-1} exists. When A_i^{-1} does not exist, Z_i and R_i are given by

$$Z_i = -b_0 \left(\frac{t^2}{2} \frac{t^3}{3!} \cdots \frac{t^{n+1}}{(n+1)!} \right), R_i = b_0 \left(t \frac{t^2}{2} \cdots \frac{t^n}{n!} \right) . \tag{33}$$

Case 1: Assume A is $n \times n$ and A^{-1} exists. Let $K = CA^{-2}$. Then

$$\begin{aligned} ZQZ^T = K \{ & Q + t (AQ + QA^T) + t^2 A Q A^T + e^{At} Q \left(e^{At}\right)^T \\ & - \left(e^{At}Q + \left[e^{At}Q\right]^T\right) - \left(te^{At}QA^T + \left[te^{At}QA^T\right]^T\right) \} K^T \end{aligned} \tag{34}$$

Integrating from $t = 0$ to Δt,

$$\begin{aligned} \int_0^{\Delta t} ZQZ^T dt = K \{ & Q\Delta t + \frac{\Delta t^2}{2} (AQ + QA^T) + \frac{\Delta t^3}{3} AQA^T \\ & \left[A^{-1} (e^{A\Delta t} - I) Q + Q \left(e^{A\Delta t} - I\right)^T A^{-T}\right] \\ & [(A^{-1}\Delta t - A^{-2}) e^{A\Delta t} + A^{-2}] QA^T - AQ \; [(A^{-1}\Delta t - A^{-2}) e^{A\Delta t} + A^{-2}]^T \\ & + \int_0^{\Delta t} e^{At} Q \left(e^{AT}\right)^T dt \; \} K^T . \end{aligned} \tag{35}$$

Let $L = CA^{-1}$, so that $R = L\,(e^{At} - I)$. Thus,

$$RQR^T = L\left\{e^{At}Q\,(e^{At})^T - \left(e^{At}Q + (e^{At}Q)^T\right) + Q\right\} L^T \tag{36}$$

and

$$\int_0^{\Delta t} RQR^T dt = L\left\{\Delta t Q - \left(A^{-1}\left[e^{A\Delta t} - I\right] Q + Q\,(e^{A\Delta t} - I)^T A^{-T}\right) + \int_0^{\Delta t} e^{At}Q\,(e^{At})^T dt\right\} L^T$$

Using the same notation,

$$\begin{aligned} ZQR^T &= K\left(I + At - e^{At}\right) Q\,(e^{At} - I)^T L^T \\ &= K\{e^{At}Q + (e^{At}Q)^T - Q - e^{At}Q\,(e^{At})^T \\ &+ AQ\,(t\,(e^{At})^T - t)\}\, L^T \end{aligned} \tag{37}$$

which implies

$$\begin{aligned} \int_0^{\Delta t} ZQR^T dt = K\{&A^{-1}\left(e^{A\Delta t} - I\right) Q + Q\left(e^{A\Delta t} - I\right)^T A^{-T} \\ &- Q\Delta t + AQ\,[(A^{-1}\Delta t - A^{-2})\,e^{A\Delta t} + A^{-2}]^T \\ &- AQ\frac{\Delta t^2}{2} - \int_0^{\Delta t} e^{At}\,Q\,(e^{At})^T dt\}\, L^T. \end{aligned} \tag{38}$$

Case 2: A^{-1} does not exist.

Given that $Q(i, j) = 0$ except when $i = j = n$, $Q(n, n) = q$, and given the form (17),(19) (section 2) for Z and R, it follows that

$$\int_0^{\Delta t} ZQZ^T dt = \frac{qb_0^2\Delta t^{2n+3}}{[(n+1)!]^2\,(2n+3)} \tag{39}$$

$$\int_0^{\Delta t} RQR^T dt = \frac{qb_0^2\Delta t^{2n+1}}{[n!]^2\,(2n+1)} \tag{40}$$

$$\int_0^{\Delta t} ZQR^T dt = \frac{-qb_0^2\Delta t^{2n+2}}{(n+1)!n!(2n+2)} \tag{41}$$

This concludes the integral of $\int_0^{\Delta t}\Psi_1(t)dt$.

Next, an expression for $\int_0^{\Delta t}\Psi_2(t)\,dt$ will be generated. $\Psi_2(t) = M_2\alpha_4 M_4^T$ is a 2 x mn matrix that is given by

$$\Psi_2(t) = \begin{bmatrix} Z_1Q_1(e^{A_1t})^T \dots\dots\dots Z_mQ_m(e^{A_mt})^T \\ R_1Q_1(e^{A_1t})^T \dots\dots\dots R_mQ_m(e^{A_mt})^T \end{bmatrix}. \tag{42}$$

For integration, first it will be assumed that A^{-1} exists.

$$ZQ\,(e^{At})^T = K\,\{Q + AQt - e^{At}Q\}\,(e^{At})^T$$

After some algebra,

$$\int_0^{\Delta t} ZQ\left(e^{At}\right)^T dt = K\left\{Q\left(e^{A\Delta t}-I\right)^T A^{-T} + AQ\left[\left(A^{-1}\Delta t - A^{-2}\right)e^{A\Delta t} + A^{-2}\right]^T - \int_0^{\Delta t} e^{At} Q\left(e^{At}\right)^T \right\} dt \tag{43}$$

Similarly, after some algebra,

$$\int_0^{\Delta t} RQ(e^{At})^T dt = L\left\{\int_0^{\Delta t} e^{At} Q\left(e^{At}\right)^T dt - Q\left(e^{A\Delta t}-I\right)^T A^{-T}\right\}$$

For the case in which A^{-1} does not exist, the (2 x n) matrix $\begin{bmatrix} ZQ\left(e^{At}\right)^T \\ RQ\left(e^{At}\right)^T \end{bmatrix}$ reduces to

$$q\begin{bmatrix} -b_0\frac{t^{n+1}}{(n+1)!} & 0 & \frac{t^{n-1}}{(n-1)!} & \frac{t^{n-2}}{(n-2)!} & \cdots\cdots t & 1 \\ 0 & b_0\frac{t^n}{n!} & \frac{t^{n-1}}{(n-1)!} & \frac{t^{n-2}}{(n-2)!} & \cdots\cdots t & 1 \end{bmatrix}. \tag{44}$$

This is a 2 x n matrix. The integral of element (1, j) is given by

$$\frac{-b_o q\ \Delta t^{2n-j+2}}{(n+1)!(n-j)!(2n-j+2)}, \tag{45}$$

while the integral of element (2, j) is given by

$$\frac{b_o q\ \Delta t^{2n-j+1}}{n!(n-j)!(2n-j+1)}, \tag{46}$$

This concludes the integral $\int_0^{\Delta t}\Psi_2(t)\,dt$, as well as the integration of $\begin{bmatrix}\Psi_1 & \Psi_2 \\ \Psi_3 & \Psi_4\end{bmatrix}(t)$.

REFERENCES

Brzezinski, A., "Polar Motion Excitation by Variations of the Effective Angular Momentum Function: Considerations Concerning Deconvolution Problem," *Manuscript Geodaetica*, 17, pp. 3-20, 1992.

Dickman, S. R., "Dynamic Ocean-tide Effects on Earth's Rotation," *J. Geophys. Res.*, *vol.* 112, pp. 448-470, 1993.

Eubanks, T. M., A. Steppe, J. O. Dickey, and P. S. Callahan, " A Spectral Analysis of the Earth's Angular Momentum Budget," *J. Geophys. Res.*, Vol. *90, pp. 5385-5404, 1985.*

Faddeeva, V. N., Computational *Methods of* Linear *Algebra*, Dover, New York, 1959.

Freedman, A. P., J. A. Steppe, J. O. Dickey, T. M. Eubanks, and L.Y. Sung, " The Short-term Prediction of Universal Time and Length of Day using Atmospheric Angular Momentum, " *J. Geophys. Res.*, Vol. 99, pp. 6981-6996, 1994.

Gross R. S., "A Combination of Earth Orientation Data: Space92," in IERS Technical Note No. 14, *Earth* orientation, *reference frames and atmospheric excitation functions submitted for the 1992 IERS Annual Report*, P. Chariot (cd.), C-1 - C-8, International Earth Rotation Service, Central Bureau of IERS-Observatoire de Paris, 1993

Hamdan, K.S., Filtering *and Prediction of Autoregressive-Moving Average Processes; Application to Earth Orientation Changes,* JPL Publication 93-14, 1993.

Haykin, S., *Adaptive Filter Theory,* Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1991.

Herring, T. A., and D. Dong, "Measurement of Diurnal and Semidiurnal Rotational Variations and Tidal Parameters of the Earth," *J. Geophys. Res.*, (in press).

Hide, R., and J. O. Dickey, "Earth's Variable Rotation," Science, Volume 253, pp. 629-637, 1991.

Keshner, M., "1/f Noise," *Proceedings of the IEEE,* Volume 70, No. 3, pp. 212-219, March 1982.

Liebelt, P. B., An *Introduction to* Optimal Estimation, Addison-Wesley Publishing Company, London, 1967.

Morabito, D. D., 'I'. M. Eubanks, and J. A. Steppe, "Kalman Filtering of Earth Orientation Changes," in *The Earth's* Rotation *and Reference Frames for Geodesy and Geodynamics,* A.K. Babcock and G. A. Wilkins (eds.), D. Reidel, Dordrecht, The Netherlands, pp. 257-267, 1988.

Priestly, M. B., *Spectral* Analysis *and Time Series,* Volume 1, Academic Press Inc., London, 1981.

Yoder, C. F. J. G. Williams, and M. E. Parke,''Tidal Variations of Earth Rotation" *J. Geophys. Res., Vol. 86, pp. 881-891, 1981.*